

Deformation of Yangian $Y(sl_2)$

S.M. Khoroshkin¹, A.A.Stolin² and V.N.Tolstoy³

¹ Institute of Theoretical & Experimental Physics
117259 Moscow, Russia (e-mail: khoroshkin@vxitep.itep.ru)

² Department of Mathematics, Royal Institute of Technology
S-10044 Stockholm, Sweden (e-mail: astolin@math.kth.se)

³ Institute of Nuclear Physics, Moscow State University
119899 Moscow, Russia (e-mail: tolstoy@anna19.npi.msu.su)

Abstract

A quantization of a non-standard rational solution of CYBE for sl_2 is given explicitly. We obtain the quantization with the help of a twisting of the usual Yangian $Y(sl_2)$. This quantum object (deformed Yangian $Y_{\eta,\xi}(sl_2)$) is a two-parametric deformation of the universal enveloping algebra $U(sl_2[u])$ of the positive current algebra $sl_2[u]$. We consider the pseudotriangular structure on $Y_{\eta,\xi}(sl_2)$, the quantum double $DY_{\eta,\xi}(sl_2)$, its the universal R-matrix and also the RTT-realization of $Y_{\eta,\xi}(sl_2)$.

0 Introduction

Yangian of $Y(g)$ of a simple Lie algebra g was introduced by V. Drinfeld [1, 2] as a deformation of the universal enveloping algebra $U(g[u])$ of the current algebra $g[u]$. Recently the Yangian symmetry $Y(sl_n)$ was shown for the following one-dimensional N-body integrable models: the Hubbard model [3, 4], the classical sl_n Euler-Calogero-Moser model confined in an external harmonic potential [5], the quantum sl_n Calogero model confined in the harmonic potential [6], and the quantum Sutherland model [7, 8]. Moreover the Yangian representation theory applied to the S-matrix theory of the $G \times G$ -invariant σ -model (G -principal chiral model) [9], and also to the system with non-local conserved currents [10] and so on.

The Yangians are an useful tool not only in the physics but in the classical representation theory of the simple Lie algebras as well. In particular Yangians are employed in [11, 12] for an explicit description of the center of the universal enveloping algebra $U(g)$, where g is a simple Lie algebra of A-, B-, C-, D-series. In other words Yangians enable one to construct new Laplace operators. A connection between the Yangian $Y(gl_2)$ and the classical construction of the Gelfand-Zetlin basis for the Lie algebra gl_2 was established in [13].

Tensor products of finite dimensional representations of the Yangian $Y(g)$ produce rational solutions of the quantum Yang-Baxter equation (QYBE). For instance, for $g = sl_2$ these solutions can be obtained by the fusion procedure applied to the Yang solution $R(u) = 1 + \mathbf{p}/u$, where \mathbf{p} is the permutation of factors in $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Another way to obtain a rational solution of QYBE is to find the image of the universal R-matrix for $Y(g)$ in a tensor product of finite dimensional representations of $Y(g)$ (see [14]). The classical r-matrix, which corresponds to the universal R-matrix for $Y(g)$ is $r = \mathbf{c}_2/u$, where \mathbf{c}_2 is

the canonical invariant element of $g \otimes g$, namely $\mathbf{c}_2 = \sum_i I_i \otimes I^i$, where $\{I_i\}$ and $\{I^i\}$ are dual bases for g with respect to the Killing form. Therefore, Yangians provide a "sophisticated" way to produce rational solutions of QYBE. However, there exist other rational solutions of the classical Yang-Baxter equation (CYBE). These solutions were studied in [15]. Every rational solution of CYBE provides a bialgebra structure on $g[u]$. These structures have not been quantized yet except the case $r = \mathbf{c}_2/u$, when the Yangian is exactly the quantization.

We present here a quantization of the simplest non-standard rational r-matrix for sl_2 , namely $r = \mathbf{c}_2/u + h_\alpha \wedge e_{-\alpha}$. Here $e_{\pm\alpha}$, h_α is the standard Chevalley basis for sl_2 . We note that the additional term $h_\alpha \wedge e_{-\alpha}$ leads to a deformation of the co-algebra structure of $Y(sl_2)$. We perform this deformation by means of the twisting of $U(sl_2) \otimes U(sl_2)$ by some special two-tensor of $U(sl_2) \otimes U(sl_2)$ (appeared in [16]). Moreover, this two-tensor enables us to write down a quantum R-matrix corresponding to the classical r-matrix $r = \mathbf{c}_2/u + h_\alpha \wedge e_{-\alpha}$. On the other hand, writing down this R-matrix explicitly in the fundamental representation, we develop RTT-formalism (see [17]) to get another presentation of the deformed Yangian. We discuss also properties of the corresponding quantum determinant, and the realization of the deformed Yangian $Y_{\eta,\xi}(sl_2)$ in terms of generating functions ("field" realization).

The existence of the quantum determinant and the field realization of the deformed Yangian can be used in a description of the center of $U(g)$ and we hope to discuss this question in future publications.

1 Non-Standard Quantization of $U(sl_2)$

Let $e_{\pm\alpha}$, h_α be the Chevalley basis for the universal enveloping algebra $U(sl_2)$ of the Lie algebra sl_2 with the standard defining relations:

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}. \quad (1.1)$$

Here and anywhere we put $(\alpha, \alpha) = 2$. Let $U(b_-) \in U(sl_2)$ be the universal enveloping algebra of the Borel subalgebra b_- of sl_2 , generated by the elements h_α and $e_{-\alpha}$. Let us introduce the following two-tensor (a formal series) F of some extension of $U(b_-) \otimes U(b_-)$:

$$F = 1 + \xi h_\alpha \otimes e_{-\alpha} + \frac{\xi^2}{2} h_\alpha (h_\alpha + 2) \otimes e_{-\alpha}^2 + \dots = \sum_{k \geq 0} \frac{\xi^k}{k!} \left(\prod_{i=0}^{k-1} (h_\alpha + 2i) \right) \otimes e_{-\alpha}^k, \quad (1.2)$$

where $\xi \in \mathbf{C}$ is some parameter. We borrowed the element F from [16]. It is not difficult to verify that the following series

$$F^{-1} = 1 - \xi h_\alpha \otimes e_{-\alpha} + \frac{\xi^2}{2} h_\alpha (h_\alpha - 2) \otimes e_{-\alpha}^2 + \dots = \sum_{k \geq 0} \frac{(-\xi)^k}{k!} \left(\prod_{i=0}^{k-1} (h_\alpha - 2i) \right) \otimes e_{-\alpha}^k \quad (1.3)$$

is a inverse element to F , i.e. $FF^{-1} = F^{-1}F = 1$. The following proposition is valid.

Proposition 1.1 *The element F (as formal series) satisfies the following relation*

$$F^{12}(\Delta \otimes \text{id})F = F^{23}(\text{id} \otimes \Delta)F, \quad (1.4)$$

where Δ is the usual comultiplication in $U(b_-)$, i.e. $\Delta(a) = a \otimes 1 + 1 \otimes a$ for any $a \in b_-$.

Proof. By direct calculations.

As consequence of this proposition we have

Corollary 1.1 *Let \mathcal{A} be an arbitrary Hopf algebra containing the Hopf algebra $U(b_-)$ and let $\bar{\mathcal{A}}_\xi^{(F)}$ be the algebra $\mathcal{A}[[\xi]]$ (i.e. the algebra \mathcal{A} over $\mathbf{C}[[\xi]]$) with co-multiplication map $\Delta^{(F)}$ given by the formula*

$$\Delta^{(F)}(a) = F\Delta(a)F^{-1}, \quad (\forall a \in \bar{\mathcal{A}}_\xi^{(F)}), \quad (1.5)$$

then $\bar{\mathcal{A}}_\xi^{(F)}$ is a Hopf algebra.

Proof. Coassociativity of $\Delta^{(F)}$ follows from the formula (1.4). Existence of the antipod is proved in [18].

Now we introduce the following notations for some elements of $\bar{U}_\xi(b_-)^{(F)}$:

$$T_\alpha := 1 - \xi e_{-\alpha}, \quad T_\alpha^{-1} := (1 - \xi e_{-\alpha})^{-1}. \quad (1.6)$$

Proposition 1.2 *The elements $h_\alpha, T_\alpha^{\pm 1}$ satisfy the following relations:*

$$T_\alpha T_\alpha^{-1} = T_\alpha^{-1} T_\alpha, \quad [h_\alpha, T_\alpha] = 2(1 - T_\alpha), \quad [h_\alpha, T_\alpha^{-1}] = 2(T_\alpha^{-1} - T_\alpha^{-2}), \quad (1.7)$$

$$\Delta^{(F)}(h_\alpha) = h_\alpha \otimes T_\alpha^{-1} + 1 \otimes h_\alpha, \quad \Delta^{(F)}(T_\alpha) = T_\alpha \otimes T_\alpha, \quad \Delta^{(F)}(T_\alpha^{-1}) = T_\alpha^{-1} \otimes T_\alpha^{-1}, \quad (1.8)$$

$$S(h_\alpha) = -h_\alpha T_\alpha, \quad S(T_\alpha) = T_\alpha^{-1}, \quad S(T_\alpha^{-1}) = T_\alpha, \quad (1.9)$$

$$\varepsilon(h_\alpha) = 0, \quad \varepsilon(T_\alpha) = \varepsilon(T_\alpha^{-1}) = 1. \quad (1.10)$$

The algebra $U_\xi(b_-)^{(F)}$ generated by $h_\alpha, T_\alpha^{\pm 1}$ is a Hopf subalgebra of $\bar{U}_\xi(b_-)^{(F)}$.

Proof. By direct calculations with (1.1), (1.4) and (1.5).

The Hopf algebra $U_\xi(b_-)^{(F)}$ is triangular with the universal R-matrix

$$\begin{aligned} R &= F^{21} F^{-1} = 1 + \xi e_{-\alpha} \wedge h_\alpha + \dots = \\ &= \sum_{k,m \geq 0} \frac{(-1)^m \xi^{k+m}}{k!m!} \left(\prod_{i=0}^{m-1} (h_\alpha + 2k - 2i) \right) e_{-\alpha}^k \otimes \left(\prod_{i=0}^{k-1} (h_\alpha + 2j) \right) e_{-\alpha}^m = \\ &= \sum_{k,m \geq 0} \frac{(-1)^m \xi^{k+m}}{k!m!} \left(\prod_{i=0}^{m-1} (h_\alpha + 2k - 2i) \right) (1 - T_\alpha)^k \otimes \left(\prod_{i=0}^{k-1} (h_\alpha + 2j) \right) (1 - T_\alpha)^m = \\ &R^{21} = R^{-1}. \end{aligned} \quad (1.11)$$

This algebra is a quantization of the bialgebra Lie b_- defined by the classical r-matrix $r = e_{-\alpha} \wedge h_\alpha$. According to Corollary 1.1 we can extend the twisting by F to $\bar{U}_\xi(sl_2)^{(F)}$. Then we have

Proposition 1.3 *Let $U_\xi(sl_2)^{(F)}$ be an algebra generated the elements h_α, e_α and $T_\alpha^{\pm 1}$ with the defining relations*

$$T_\alpha T_\alpha^{-1} = T_\alpha^{-1} T_\alpha, \quad [h_\alpha, T_\alpha] = 2(1 - T_\alpha), \quad [h_\alpha, T_\alpha^{-1}] = 2(T_\alpha^{-1} - T_\alpha^{-2}), \quad (1.13)$$

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [T_\alpha, e_\alpha] = 2\xi h_\alpha, \quad [T_\alpha^{-1}, e_\alpha] = -2\xi T_\alpha^{-1} h_\alpha T_\alpha^{-1}, \quad (1.14)$$

$$\Delta^{(F)}(h_\alpha) = h_\alpha \otimes T_\alpha^{-1} + 1 \otimes h_\alpha, \quad \Delta^{(F)}(T_\alpha) = T_\alpha \otimes T_\alpha, \quad \Delta^{(F)}(T_\alpha^{-1}) = T_\alpha^{-1} \otimes T_\alpha^{-1}, \quad (1.15)$$

$$\Delta^{(F)}(e_\alpha) = e_\alpha \otimes T_\alpha^{-1} + 1 \otimes e_\alpha - \xi h_\alpha \otimes T_\alpha^{-1} h_\alpha - \frac{\xi}{2} h_\alpha (h_\alpha - 2) \otimes T_\alpha^{-1} - \frac{\xi}{2} h_\alpha (h_\alpha + 2) \otimes T_\alpha^{-2}, \quad (1.16)$$

$$S(h_\alpha) = -h_\alpha T_\alpha, \quad S(T_\alpha) = T_\alpha^{-1}, \quad S(T_\alpha^{-1}) = T_\alpha, \quad (1.17)$$

$$S(e_\alpha) = -e_\alpha T_\alpha - \frac{\xi}{2} h_\alpha (h_\alpha + 2) T_\alpha (T_\alpha - 2), \quad (1.18)$$

$$\varepsilon(h_\alpha) = \varepsilon(e_\alpha) = 0, \quad \varepsilon(T_\alpha) = \varepsilon(T_\alpha^{-1}) = 1. \quad (1.19)$$

Then $U_\xi(sl_2)^{(F)}$ is a Hopf subalgebra of $\bar{U}_\xi(sl_2)^{(F)}$ and it is a triangular deformation of $U(sl_2)$ in the direction of the classical r -matrix $r = e_{-\alpha} \wedge h_\alpha$.

Proof. By direct calculations with (1.1) and (1.5).

Remark 1. In every finite dimensional representation of sl_2 the element $T_\alpha = 1 - 2\xi e_{-\alpha}$ is always invertible since $e_{-\alpha}$ is nilpotent. Therefore, the theory of finite dimensional representations of $U_\xi(sl_2)^{(F)}$ is the same as the theory for sl_2 .

Remark 2. Similar computations for sl_2 with another twisting element \tilde{F} were carried out in [21]. However, using Theorem 2 from [19] one can prove the following result.

Theorem 1.1 *There exists an invertible element $T \in U(gl_2)[[\xi]]$ such that $\tilde{F} = (T \otimes T) \cdot F^{-1} \cdot \Delta(T^{-1})$, $\varepsilon(T) = 1$ and $\rho_1(T) = 1 \in GL_2$, where ρ_1 is the two-dimensional representation of gl_2 . In other words Hopf algebras obtained from $U(sl_2)[[\xi]]$ by twisting by \tilde{F}^{-1} and F are isomorphic as Hopf algebras and the isomorphism is given by conjugation by T .*

Proof. The matrix $(\rho_1 \otimes \rho_1)((\tilde{F}^{21})^{-1} \tilde{F})$ was computed in [21]. We can calculate the matrix $(\rho_1 \otimes \rho_1)(F^{21} F^{-1})$ and it turns out that we obtain the same matrix (see Lemma 5.1 further). Then Theorem 2 from [19] implies all the statements.

2 Rational solutions of CYBE

The theory of rational solutions of CYBE over any simple Lie algebra g was developed in [15] and we would like to remind several basic facts of the theory.

Let $P(u, v) = \frac{\mathbf{c}_2}{u-v} + r(u, v)$ be a function from \mathbf{C}^2 to $g \otimes g$, where $r(u, v)$ is a polynomial in u, v . If $P(u, v)$ satisfies CYBE, we say that $P(u, v)$ is a rational solution of CYBE. It is possible to show that every rational solution can be brought by means of a gauge transformation to the form

$$P(u, v) = \frac{\mathbf{c}_2}{u-v} + r_{00} + r_{10}u + r_{01}v + r_{11}uv, \quad (2.1)$$

where $r_{00}, r_{10}, r_{01}, r_{11} \in g \otimes g$. Every rational solution induces a bialgebra structure on the current algebra $g[t]$. It is possible to prove the following theorems.

Theorem 2.1 *Let $D(g[t])$ be the classical double corresponding to a rational solution of CYBE. Then $D(g[t])$ and $g((t^{-1}))$ are isomorphic as Lie algebras with inner product which takes the following form on $g((t^{-1}))$:*

$$(a(t), b(t)) = \text{Res}_{t=0} \langle a(t), b(t) \rangle. \quad (2.2)$$

where $a(t), b(t) \in g((t^{-1}))$. It means that $g((t^{-1}))$ can be represented as a Manin triple $g((t^{-1})) = g[t] \oplus \mathcal{W}$ where \mathcal{W} is a Lagrangian subalgebra with respect to the inner product (2.2).

Theorem 2.2 Let $P(u, v) = \frac{\mathbf{c}_2}{u-v} + r_{00}$, where $r_{00} \in g \otimes g$. Then the following conditions are equivalent:

- (i) the function $P(u, v)$ is a rational r -matrix;
- (ii) the element $r_{0,0}$ satisfies CYBE;
- (iii) the Lagrangian subalgebra \mathcal{W} is contained in $g[[t^{-1}]]$ (this is equivalent to $t^{-2}g[[t^{-1}]] \subset \mathcal{W} \subset g[[t^{-1}]]$).

Now it is possible to deduce that rational solutions satisfying the Theorem (2.2) are in a 1-1 correspondence with the following combinatorial data:

- 1) subalgebra \mathcal{L} of g ;
- 2) non-degenerate 2-cocycle B on \mathcal{L} .

In case of $g = sl_n$ all the rational solutions can be described in a similar way. Let

$$d_k = \text{diag}(\underbrace{1, \dots, 1}_k, t, \dots, t) \in GL(n, \mathbf{C}((t^{-1}))) . \quad (2.3)$$

Then every rational solution of CYBE defines some Lagrangian subalgebra \mathcal{W} contained in $d_k^{-1} \cdot sl(n, \mathbf{C}[[t^{-1}]]) \cdot d_k$ for some k . The corresponding combinatorial data are:

- (1) subalgebra $\mathcal{L} \subset sl(n, \mathbf{C})$ such that $\mathcal{L} + \mathcal{P}_k = sl(n, \mathbf{C})$, where \mathcal{P}_k is the maximal parabolic subalgebra of $sl(n, \mathbf{C})$ not containing the root vector e_{α_k} of the simple root α_k ;
- (2) 2-cocycle B on \mathcal{L} which is nondegenerate on $\mathcal{L} \cap \mathcal{P}_k$.

In case of sl_2 one has just two non-standard rational r -matrices (up to gauge equivalence):

$$P_1(u, v) = \frac{\mathbf{c}_2}{u-v} + h_\alpha \wedge e_{-\alpha} , \quad (2.4)$$

and

$$P_2(u, v) = \frac{\mathbf{c}_2}{u-v} + h_\alpha \otimes e_{-\alpha}u - e_{-\alpha} \otimes h_\alpha v . \quad (2.5)$$

The corresponding Lagrangian subalgebras are

$$\mathcal{W}_1 = t^{-2}(sl_2[[t^{-1}]]) \oplus \mathbf{C} \cdot (e_\alpha t^{-1} - h_\alpha) \oplus \mathbf{C}(e_{-\alpha} t^{-1}) \oplus \mathbf{C}(h_\alpha t^{-1} + 2e_{-\alpha}) , \quad (2.6)$$

and

$$\begin{aligned} \mathcal{W}_2 = & t^{-3}(sl_2[[t^{-1}]] \oplus \mathbf{C}(e_\alpha t^{-1} - h_\alpha t) \oplus \mathbf{C}(e_{-\alpha} t^{-1}) \oplus \\ & \oplus \mathbf{C}(h_\alpha t^{-1}) \oplus \mathbf{C}(e_\alpha t^{-2}) \oplus \mathbf{C}(e_{-\alpha} t^{-2}) \oplus \mathbf{C}(h_\alpha t^{-2} + 2e_{-\alpha}) . \end{aligned} \quad (2.7)$$

Now we are going to quantize the rational r -matrix (2.4).

3 Twisting of Yangian $Y(sl_2)$

One can show that the Yangian $Y(sl_2)$ (as a Hopf algebra) can be defined by Chevalley generators $h_\alpha, e_{\pm\alpha}, e_{\delta-\alpha}$ with the defining relations [14]:

$$[e_\alpha, e_{-\alpha}] = h_\alpha , \quad [h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha} , \quad (3.1)$$

$$[h_\alpha, e_{\delta-\alpha}] = e_{\delta-\alpha} , \quad [e_{-\alpha}, e_{\delta-\alpha}] = \eta e_{-\alpha} , \quad (3.2)$$

$$[e_\alpha, [e_\alpha, [e_\alpha, e_{\delta-\alpha}]]] = 6\eta e_\alpha^2, \quad [[e_\alpha, e_{\delta-\alpha}], e_{\delta-\alpha}], e_{\delta-\alpha}] = 6\eta e_{\delta-\alpha}^2. \quad (3.3)$$

$$\Delta(h_\alpha) = h_\alpha \otimes 1 + 1 \otimes h_\alpha, \quad \Delta(e_{\pm\alpha}) = e_{\pm\alpha} \otimes 1 + 1 \otimes e_{\pm\alpha}, \quad (3.4)$$

$$\Delta(e_{\delta-\alpha}) = e_{\delta-\alpha} \otimes 1 + 1 \otimes e_{\delta-\alpha} + \eta e_\alpha \otimes h_\alpha, \quad (3.5)$$

$$S(h_\alpha) = -h_\alpha, \quad S(e_{\pm\alpha}) = -e_{\pm\alpha}, \quad S(e_{\delta-\alpha}) = -e_{\delta-\alpha} - \eta e_{-\alpha} h_\alpha, \quad (3.6)$$

$$\varepsilon(h_\alpha) = \varepsilon(e_{\pm\alpha}) = \varepsilon(e_{\delta-\alpha}) = 0, \quad \varepsilon(1) = 1. \quad (3.7)$$

Here we explicitly introduce the Yangian deformation parameter $\eta \in \mathbf{C} \setminus \{0\}^1$ and therefore we shall use further the notation $Y_\eta(sl_2)$ for the Yangian.

Since $Y_\eta(sl_2)$ contains $U(sl_2)$ as a Hopf subalgebra the Corollary 1.1 implies that the algebra $\bar{Y}_{\eta,\xi}^{(F)}$ isomorphic to $Y_\eta[[\xi]]$ with the comultiplication (1.5) is a Hopf algebra.

Proposition 3.1 *The elements $h_\alpha, e_\alpha, T_\alpha^{\pm 1}$ (see (1.6)) and $e_{\delta-\alpha}$ satisfy the relations:*

$$T_\alpha T_\alpha^{-1} = T_\alpha^{-1} T_\alpha, \quad [h_\alpha, T_\alpha] = 2(1 - T_\alpha), \quad [h_\alpha, T_\alpha^{-1}] = 2(T_\alpha^{-1} - T_\alpha^{-2}), \quad (3.8)$$

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [T_\alpha, e_\alpha] = 2\xi h_\alpha, \quad [T_\alpha^{-1}, e_\alpha] = -2\xi T_\alpha^{-1} h_\alpha T_\alpha^{-1}, \quad (3.9)$$

$$[T_\alpha, e_{\delta-\alpha}] = -\frac{\eta}{2\xi} (T_\alpha^2 - 2T_\alpha + 1), \quad [T_\alpha^{-1}, e_{\delta-\alpha}] = -\frac{\eta}{2\xi} (T_\alpha^{-2} - 2T_\alpha^{-1} + 1), \quad (3.10)$$

$$[e_\alpha, [e_\alpha, [e_\alpha, e_{\delta-\alpha}]]] = 6\eta e_\alpha^2, \quad [[e_\alpha, e_{\delta-\alpha}], e_{\delta-\alpha}], e_{\delta-\alpha}] = 6\eta e_{\delta-\alpha}^2. \quad (3.11)$$

$$\Delta^{(F)}(h_\alpha) = h_\alpha \otimes T_\alpha^{-1} + 1 \otimes h_\alpha, \quad \Delta^{(F)}(T_\alpha) = T_\alpha \otimes T_\alpha, \quad \Delta^{(F)}(T_\alpha^{-1}) = T_\alpha^{-1} \otimes T_\alpha^{-1}, \quad (3.12)$$

$$\Delta^{(F)}(e_\alpha) = e_\alpha \otimes T_\alpha^{-1} + 1 \otimes e_\alpha - \xi h_\alpha \otimes T_\alpha^{-1} h_\alpha - \frac{\xi}{2} h_\alpha (h_\alpha - 2) \otimes T_\alpha^{-1} - \frac{\xi}{2} h_\alpha (h_\alpha + 2) \otimes T_\alpha^{-2}, \quad (3.13)$$

$$\Delta^{(F)}(e_{\delta-\alpha}) = e_\alpha \otimes T_\alpha + 1 \otimes e_{\delta-\alpha} + \xi h_\alpha \otimes T_\alpha^{-1} + \frac{\eta}{2\xi} h_\alpha \otimes (1 - T_\alpha), \quad (3.14)$$

$$S(h_\alpha) = -h_\alpha T_\alpha, \quad S(T_\alpha) = T_\alpha^{-1}, \quad S(T_\alpha^{-1}) = T_\alpha, \quad (3.15)$$

$$S(e_\alpha) = -e_\alpha T_\alpha - \frac{\xi}{2} h_\alpha (h_\alpha + 2) T_\alpha (T_\alpha - 1), \quad (3.16)$$

$$S(e_{\delta-\alpha}) = -e_{\delta-\alpha} T_\alpha^{-1} - \frac{\xi}{\eta} h_\alpha T_\alpha + \frac{\eta}{2\xi} h_\alpha T_\alpha^{-1} - \frac{\eta}{2\xi} h_\alpha, \quad (3.17)$$

$$\varepsilon(h_\alpha) = \varepsilon(e_\alpha) = \varepsilon(e_{\delta-\alpha}) = 0, \quad \varepsilon(T_\alpha) = \varepsilon(T_\alpha^{-1}) = 1. \quad (3.18)$$

The algebra $Y_{\eta,\xi}(sl_2)$ generated the elements $h_\alpha, e_\alpha, T_\alpha^{\pm 1}, e_{\delta-\alpha}$ is a Hopf subalgebra of $\bar{Y}_{\eta,\xi}(sl_2)^{(F)}$.

Proof. By direct calculations.

One can see that the Hopf algebra $Y_{\eta,\xi}(sl_2)^{(F)}$ is a quantization of the Lie bialgebra $sl_2[u]$ corresponding to the rational solution (2.4).

¹All Yangians with different η are isomorphic one to another and therefore one usually suppose that $\eta = 1$.

4 Pseudotriangular structure on $Y_{\eta,\xi}(sl_2)$. Deformed Yangian double $DY_\eta(sl_2)$

We want to recall that $Y_\eta(sl_2)$ is a pseudotriangular Hopf algebra [1]. It means that there is a family of automorphisms $\mathcal{T}_\lambda : Y_\eta(sl_2) \mapsto Y_\eta(sl_2)$ and an element $R(\lambda) = 1 + \sum_{k=1}^\infty R_k \lambda^{-1}$, where $R_k \in Y_\eta(sl_2) \otimes Y_\eta(sl_2)$, such that

$$(\mathcal{T}_\lambda \otimes \mathcal{T}_\mu)R(u) = R(u + \lambda - \mu) , \quad (\mathcal{T}_\lambda \otimes \text{id})\Delta(a)' = R(\lambda)((\mathcal{T}_\lambda \otimes \text{id})\Delta(a))R(\lambda)^{-1} , \quad (4.1)$$

$$(\Delta \otimes \text{id})R(\lambda) = R^{13}(\lambda)R^{23}(\lambda) , \quad R^{12}(\lambda)R^{21}(-\lambda) = 1 \otimes 1 , \quad (4.2)$$

$$R^{12}(\lambda_1 - \lambda_2)R^{13}(\lambda_1 - \lambda_3)R^{23}(\lambda_2 - \lambda_3) = R^{23}(\lambda_2 - \lambda_3)R^{13}(\lambda_1 - \lambda_3)R^{12}(\lambda_1 - \lambda_2) . \quad (4.3)$$

As was proved in [22], $\bar{Y}_{\eta,\xi}(sl_2)^{(F)}$ is also a pseudotriangular Hopf algebra with respect to $\Delta^{(F)} = F\Delta F^{-1}$, $R(\lambda)^{(F)} = F^{21}R(\lambda)F^{-1}$ and the same \mathcal{T}_λ .

Since $Y_\eta(sl_2) \subset Y_{\eta,\xi}(sl_2) \subset \bar{Y}_{\eta,\xi}(sl_2)^{(F)}$ (as associative algebras) and since \mathcal{T}_λ acts identically on $U(sl_2)[[\xi]]$ and since $Y_{\eta,\xi}(sl_2)$ differs from $Y_\eta(sl_2)$ by elements from $U(sl_2)[[\xi]]$ therefore the automorphisms \mathcal{T}_λ extended to $\bar{Y}_{\eta,\xi}(sl_2)^{(F)}$ preserve $Y_{\eta,\xi}(sl_2)$. Thus we have the following result.

Proposition 4.1 *The Hopf algebra $Y_{\eta,\xi}(sl_2)$ is pseudotriangular with $R(\lambda)^{(F)} := F^{21}R(\lambda)F^{-1}$. In particular, $R(\lambda)^{(F)}$ is a rational solution of QYBE.*

Let us recall that the Yangian double $DY_\eta(sl_2)$ (see [14]) is a quasitriangular Hopf algebra with an universal R-matrix \mathcal{R} which lies in some extension of $DY_\eta(sl_2) \otimes DY_\eta(sl_2)$. Since $U(sl_2) \subset DY_\eta(sl_2)$ we can twist the Yangian double $DY_\eta(sl_2)$ by F . Using formal algebraic arguments similar to that of [22] we get as result the following proposition.

Proposition 4.2 *The deformed Yangian double $DY_{\eta,\xi}(sl_2)^{(F)}$ is a quasitriangular Hopf algebra with the universal R-matrix $\mathcal{R}^{(F)} = F^{21}\mathcal{R}F^{-1}$.*

In what follows we need the realization of the Yangian double $DY_\eta(sl_2)$ given in [14]. In this realization the Yangian double $DY_\eta(sl_2)$ is generated by the elements $h_{k\delta}$, $e_{k\delta \pm \alpha}$, ($k \in \mathbf{Z}$)², which are composed into generating functions $h^+(u) = 1 + \sum_{k \geq 0} h_{k\delta} u^{-k-1}$, ($h_0 := h_\alpha$), $e_{\pm\alpha}^+(u) = \sum_{k \geq 0} e_{k\delta \pm \alpha} u^{-k-1}$, and $h^-(u) = 1 - \sum_{k < 0} h_{k\delta} u^{-k-1}$, $e_{\pm\alpha}^-(u) = -\sum_{k < 0} e_{k\delta \pm \alpha} u^{-k-1}$, which satisfy the following relations:

$$[h^\pm(u), h^\pm(v)] = 0 , \quad [h^+(u), h^-(v)] = 0 , \quad (4.4)$$

$$[e_\alpha^\pm(u), e_{-\alpha}^\pm(v)] = -\eta \frac{h^\pm(u) - h^\pm(v)}{u - v} , \quad (4.5)$$

$$[e_\alpha^\pm(u), e_{-\alpha}^\mp(v)] = -\eta \frac{h^\mp(u) - h^\pm(v)}{u - v} , \quad (4.6)$$

$$[h^\pm(u), e_{\pm\alpha}^\pm(v)] = -\eta \frac{\{h^\pm(u), (e_{\pm\alpha}^\pm(u) - e_{\pm\alpha}^\pm(v))\}}{u - v} , \quad (4.7)$$

$$[h^\pm(u), e_{\pm\alpha}^\mp(v)] = -\eta \frac{\{h^\pm(u), (e_{\pm\alpha}^\pm(u) - e_{\pm\alpha}^\mp(v))\}}{u - v} , \quad (4.8)$$

²These notations of the generators are connected with the notations in [14] as follows: $h_{k\delta} := h_k$, $e_{k\delta + \alpha} := e_k$, $e_{k\delta - \alpha} := f_k$ ($k \in \mathbf{Z}$).

$$[e_{\pm\alpha}^{\pm}(u), e_{\pm\alpha}^{\pm}(v)] = \mp\eta \frac{(e_{\pm\alpha}^{\pm}(u) - e_{\pm\alpha}^{\pm}(v))^2}{u - v}, \quad (4.9)$$

$$[e_{\pm\alpha}^{+}(u), e_{\pm\alpha}^{-}(v)] = \mp\eta \frac{(e^{+}(u) - e^{-}(v))^2}{u - v}, \quad (4.10)$$

where $\{a, b\} = ab + ba$. It turns out that $Y_{\eta}(sl_2)$ is generated by $h^{+}(u)$, $e_{\pm\alpha}^{+}(u)$, while the dual to $Y_{\eta}(sl_2)$ algebra $Y_{\eta}^{\circ}(sl_2)$ is generated by $h^{-}(u)$, $e_{\pm\alpha}^{-}(u)$. The universal R-matrix \mathcal{R} was found in [14] can be factorized as follows:

$$\mathcal{R} = \mathcal{R}_{+} \mathcal{R}_0 \mathcal{R}_{-}, \quad (4.11)$$

where

$$\mathcal{R}_{+} = \prod_{k \geq 0}^{\rightarrow} \exp(-e_{-(k+1)\delta-\alpha} \otimes e_{k\delta+\alpha}), \quad \mathcal{R}_{-} = \prod_{k \geq 0}^{\leftarrow} \exp(-e_{-(k+1)\delta+\alpha} \otimes e_{k\delta-\alpha}), \quad (4.12)$$

$$\mathcal{R}_0 = \prod_{n \geq 0} \exp \text{Res}_{u=v} \left(\ln h^{-}(v + 2n + 1) \otimes \frac{d}{du} \ln h^{+}(u) \right). \quad (4.13)$$

Here $\text{Res}_{u=v}(f(u) \otimes g(v)) = \sum_k f_k \otimes g_{-k-1}$ if $f(u) = \sum f_k u^{-k-1}$, $g(v) = \sum g_k v^{-k-1}$.

Corollary 4.1 *The element $\mathcal{R}^{(F)} = F\mathcal{R}F^{-1}$ satisfies QYBE, where F is the same as in (1.2) and \mathcal{R} is defined by (4.11)-(4.13).*

5 RTT-realization of the deformed Yangian $Y_{\eta,\xi}(sl_2)$

In the section we develop so called RTT-formalism (see [17]), i.e. we obtain the RTT-realization or in other words the realization in terms of the L-operator.

Let $\rho^{(1)}$ be the two-dimensional representation of sl_2 in \mathbf{C}^2 with the basis $|1\rangle$ and $|-1\rangle$. It is well-known that $\rho^{(1)}$ is extended to a representation $\rho_u^{(1)}$ ($u \in \mathbf{C}$) of the Yangian $Y_{\eta}(sl_2)$ by means of

$$\begin{aligned} \rho_u^{(1)}(h^{\pm}(w)) |1\rangle &= \frac{\eta}{w-u} |1\rangle, & \rho_u^{(1)}(h^{\pm}(w)) |-1\rangle &= \frac{-\eta}{w-u} |-1\rangle, \\ \rho_u^{(1)}(e_{\alpha}^{\pm}(w)) |1\rangle &= 0, & \rho_u^{(1)}(e_{\alpha}^{\pm}(w)) |-1\rangle &= \frac{\eta}{w-u} |1\rangle, \\ \rho_u^{(1)}(e_{-\alpha}^{\pm}(w)) |1\rangle &= \frac{\eta}{w-u} |-1\rangle, & \rho_u^{(1)}(e_{-\alpha}^{\pm}(w)) |-1\rangle &= 0, \end{aligned} \quad (5.1)$$

With the help of these formulas we find

$$(\rho_u^{(1)} \otimes \rho_v^{(1)})(\mathcal{R}) = \varphi(u-v) \left(1 + \frac{\mathbf{P}_{12}}{u-v} \right) = \varphi(u-v) R(u-v), \quad (5.2)$$

where φ is a scalar function and \mathbf{p}_{12} interchanges factors in $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Let $L(u) = (\rho_u^{(1)} \otimes \text{id})(\mathcal{R})$ then QYBE for $R(u-v)$ implies that

$$R(u-v) \overset{1}{L}(u) \overset{2}{L}(v) = \overset{2}{L}(v) \overset{1}{L}(u) R(u-v), \quad (5.3)$$

where $\overset{1}{L}(u) = L(u) \otimes \text{id}$, $\overset{2}{L}(v) = \text{id} \otimes L(v)$. The matrix $L(u)$ is a generating function for $Y_{\eta}(gl_2)$ and $Y(sl_2) \cong Y_{\eta}(gl_2)/(\text{qdet} L(u) - 1)$. More exactly we can formulate the following result.

Proposition 5.1 *Let $L(u)$ be a 2×2 -matrix with noncommuting entries, such that*

- (i) $R(u-v) \overset{1}{L}(u) \overset{2}{L}(v) = \overset{2}{L}(v) \overset{1}{L}(u) R(u-v)$,
 - (ii) $L(u) = 1 + \frac{L(0)}{u} + \frac{L(1)}{u^2} + \dots \frac{L(k)}{u^k} + \dots$,
 - (iii) $\text{qdet } L(u) = e_{11}(u)e_{22}(u-1) - e_{21}(u)e_{12}(u-1) = e_{22}(u)e_{11}(u-1) - e_{12}(u)e_{21}(u-1) = 1$.
- Then the matrix coefficients of $L(u)$ generate a Hopf algebra isomorphic to $Y_\eta(sl_2)$. The comultiplication Δ and the antipode S are given by the formulas*

$$\Delta(e_{ij}(u)) = \sum_k e_{ik}(u) \otimes e_{kj}(u) . \quad (5.4)$$

$$S(L(u)) = L^{-1}(u) . \quad (5.5)$$

The deformed Yangian $Y_{\eta,\xi}(sl_2)$ admits a similar representation. We start with the lemma.

Lemma 5.1 *In the representation $\rho_u^{(1)} \otimes \rho_v^{(1)}$ the universal R -matrix $\mathcal{R}^{(F)} = F^{21}\mathcal{R}F^{-1}$ has the form*

$$\begin{aligned} R_{\eta,\xi}(u-v) &:= (\rho_u^{(1)} \otimes \rho_v^{(1)})(\mathcal{R}^{(F)}) = \\ &= \left(1 + \xi \rho_u^{(1)}(e_{-\alpha}) \otimes \rho_v^{(1)}(h_\alpha)\right) \left(1 - \eta \frac{\mathbf{P}_{12}}{u-v}\right) \left(1 - \xi \rho_u^{(1)}(h_\alpha) \otimes \rho_v^{(1)}(e_{-\alpha})\right) = \\ &= \begin{pmatrix} 1 - \frac{\eta}{u-v} & 0 & 0 & 0 \\ -\xi & 1 & -\frac{\eta}{u-v} & 0 \\ \xi & -\frac{\eta}{u-v} & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 - \frac{\eta}{u-v} \end{pmatrix} \end{aligned} \quad (5.6)$$

Proof. By direct calculation.

Let us consider an algebra A of matrix elements of $L(u)$ satisfying the relation

$$R_{\eta,\xi}(u-v) \overset{1}{L}(u) \overset{2}{L}(v) = \overset{2}{L}(v) \overset{1}{L}(u) R_{\eta,\xi}(u-v) . \quad (5.7)$$

It follows that $L(u) = (\rho_u^{(1)} \otimes \text{id})\mathcal{R}^{(F)}$ satisfies (5.7). Algebra A together with the comultiplication (5.4) and antipod (5.5) constitutes a Hopf algebra. The following lemma takes place.

Lemma 5.2 (i) *The matrix $R_{\eta,\xi}(\eta)$ is the projector onto the one-dimensional subspace $\mathbf{C}(|1\rangle \otimes |-1\rangle - |-1\rangle \otimes |1\rangle - \xi|-1\rangle \otimes |-1\rangle)$ up to a scalar factor.*

(ii) *The following relation hold*

$$R_{\eta,\xi}(\eta) \overset{1}{L}(u) \overset{2}{L}(u-\eta) = \overset{2}{L}(u-\eta) \overset{1}{L}(u) R_{\eta,\xi}(\eta) = \left(\text{qdet}_{\eta,\xi} L(u)\right) R_{\eta,\xi}(\eta) , \quad (5.8)$$

$$\begin{aligned} \text{qdet}_{\eta,\xi} L(u) &= e_{11}(u)e_{22}(u-\eta) - e_{21}(u)e_{12}(u-\eta) - \xi e_{11}(u)e_{12}(u-\eta) = \\ &= e_{22}(u)e_{11}(u-\eta) - e_{12}(u)e_{21}(u-\eta) + \xi e_{12}(u)e_{11}(u-\eta) , \end{aligned} \quad (5.9)$$

$$\Delta^{(F)}(\text{qdet}_{\eta,\xi} L(u)) = \text{qdet}_{\eta,\xi} L(u) \otimes \text{qdet}_{\eta,\xi} L(u) \quad (5.10)$$

where the quantum determinant $\text{qdet}_{\eta,\xi} L(u)$ is an element of the Hopf algebra A .

Proof. The part (i) is verified by direct calculations. The proof of the second part is standard (see [11]).

Lemma 5.3 *The quantum determinant $\text{qdet}_{\eta,\xi} L(u)$ is a central element of the algebra A .*

Proof. The formula for the quantum determinant $\text{qdet}_{\eta,\xi}$ and the quantum Yang-Baxter equation for $R_{\eta,\xi}(u)$ provide the following equality:

$$\begin{aligned} & R_{\eta,\xi}^{23}(\eta) R_{\eta,\xi}^{12}(u) R_{\eta,\xi}^{13}(u + \eta) R_{\eta,\xi}^{23}(\eta) \overset{1}{L}(v) (\text{qdet}_{\eta,\xi} L(u)) R_{\eta,\xi}^{23}(\eta) = \\ & = (\text{qdet}_{\eta,\xi} L(u)) R_{\eta,\xi}^{23}(\eta) \overset{1}{L}(v) R_{\eta,\xi}^{23}(\eta) R_{\eta,\xi}^{12}(u) R_{\eta,\xi}^{13}(u + \eta) R_{\eta,\xi}^{23}(\eta) , \end{aligned}$$

where we use the standard notations: $\overset{1}{L}(u) := L(u) \otimes 1 \otimes 1$, and $R_{\eta,\xi}^{12} = \sum_i a_i \otimes b_i \otimes 1$ and so on if $R_{\eta,\xi} = \sum_i a_i \otimes b_i$. Direct calculations show that

$$R_{\eta,\xi}^{23}(\eta) R_{\eta,\xi}^{12}(u) R_{\eta,\xi}^{13}(u + \eta) R_{\eta,\xi}^{23}(\eta) = \frac{2(u - \eta)}{\eta} R_{\eta,\xi}^{23} ,$$

Therefore

$$R_{\eta,\xi}^{23}(\eta) L^1(v) (\text{qdet}_{\eta,\xi} L(u)) R_{\eta,\xi}^{23}(\eta) = R_{\eta,\xi}^{23}(\eta) (\text{qdet}_{\eta,\xi} L(u)) L^1(v) R_{\eta,\xi}^{23}(\eta) ,$$

and $\text{qdet}_{\eta,\xi} L(u)$ commutes with $L(v)$, i.e.

$$[\text{qdet}_{\eta,\xi} L(u), L(v)] = 0$$

The proof is complete.

At last we have the following theorem.

Theorem 5.1 *Let A be a Hopf algebra generated by matrix elements of the generating function $L(u)$ satisfying the following conditions:*

$$R_{\eta,\xi}(u - v) \cdot \overset{1}{L}(u) \overset{2}{L}(v) = \overset{2}{L}(v) \overset{1}{L}(u) \cdot R_{\eta,\xi}(u - v) , \quad (5.11)$$

$$L(u) = \begin{pmatrix} T_\alpha^{\frac{1}{2}} & 0 \\ \xi h_\alpha T_\alpha^{\frac{1}{2}} & T^{-\frac{1}{2}} \end{pmatrix} + \frac{L(0)}{u} + \frac{L(1)}{u^2} + \dots , \quad (5.12)$$

$$\text{qdet}_{\eta,\xi} L(u) = 1 \quad (5.13)$$

$$\Delta(e_{ij}(u)) = \sum_k e_{ik}(u) \otimes e_{kj}(u) , \quad (5.14)$$

$$S(L(u)) = L^{-1}(u) , \quad (5.15)$$

where $T_\alpha := 1 - 2\xi e_{-\alpha}$. Then A is isomorphic to $Y_{\eta,\xi}(sl_2)$.

Proof. The proof is analogous to that of Proposition 5.1. Taking into account that

$$L(u) = (\rho^{(1)}(u) \otimes \text{id})(\mathcal{R}^{(F)})$$

we see that there exists a homomorphism from A to $Y_{\eta,\xi}(sl_2)$. To see that this is an epimorphism, one can use the formula (6.2). To prove that this homomorphism is a monomorphism one can use the same arguments as for non-deformed Yangians (see for instance [23]).

Remark. To find the constant term in the decomposition of $L(u)$ into the series in u^{-1} , one should find

$$\lim_{u \rightarrow \infty} (\rho^{(1)}(u) \otimes \text{id})(\mathcal{R}^{(F)}) = (\rho^{(1)} \otimes \text{id})(\mathcal{R}^{(F)}) = \begin{pmatrix} 1 & 0 \\ \xi h_\alpha & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}, \quad (5.16)$$

where

$$z = 1 + \xi e_{-\alpha} + \frac{3!!}{2!} \xi^2 e_{-\alpha}^2 + \cdots + \frac{(2n-1)!!}{n!} \xi^n e_{-\alpha}^n + \cdots. \quad (5.17)$$

It is not difficult to see that $z^2 = T_\alpha^{-1}$, hence

$$\lim_{u \rightarrow \infty} (\rho^{(1)}(u) \otimes \text{id})(\mathcal{R}^{(F)}) = \begin{pmatrix} T_\alpha^{\frac{1}{2}} & 0 \\ \xi h_\alpha T_\alpha^{\frac{1}{2}} & T_\alpha^{-\frac{1}{2}} \end{pmatrix}. \quad (5.18)$$

6 Realization of the deformed Yangian $Y_{\eta,\xi}(sl_2)$ in terms of generating functions

The realization of the usual Yangian $Y_\eta(sl_2)$ in terms of the generating functions ("fields" realization) $h_\alpha^+(u)$ and $e_{\pm\alpha}^+(u)$ (see Section 5) can be obtained from the Gauss decomposition of the L-operator:

$$L(u) = (\rho^{(1)}(u) \otimes \text{id})(\mathcal{R}_+ \mathcal{R}_0 \mathcal{R}_-) = \begin{pmatrix} 1 & 0 \\ -e_\alpha^+(u) & 1 \end{pmatrix} \begin{pmatrix} k_1(u) & 0 \\ 0 & k_2(u) \end{pmatrix} \begin{pmatrix} 1 & -e_{-\alpha}^+(u) \\ 0 & 1 \end{pmatrix}. \quad (6.1)$$

where $k_1(u)k_2(u-1) = 1$, $k_2(u)k_1^{-1}(u) = h^+(u)$ (see [14] and [23]).

The same procedure can be applied to $\mathcal{R}^{(F)}$. We have

$$\begin{aligned} L(u) &= (\rho^{(1)}(u) \otimes \text{id})(F^{21} \mathcal{R}_+ \mathcal{R}_0 \mathcal{R}_- F^{-1}) = \\ &= \begin{pmatrix} 1 & 0 \\ \xi h_\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e_\alpha^+(u) & 1 \end{pmatrix} \begin{pmatrix} k_1(u) & 0 \\ 0 & k_2(u) \end{pmatrix} \begin{pmatrix} 1 & -e_{-\alpha}^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_\alpha^{\frac{1}{2}} & 0 \\ 0 & T_\alpha^{-\frac{1}{2}} \end{pmatrix} = \end{aligned} \quad (6.2)$$

$$= \begin{pmatrix} 1 & 0 \\ \xi h_\alpha - e_\alpha^+(u) & 1 \end{pmatrix} \begin{pmatrix} k_1(u) T_\alpha^{\frac{1}{2}} & 0 \\ 0 & k_2(u) T_\alpha^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -T_\alpha^{-\frac{1}{2}} \\ 0 & 1 \end{pmatrix}. \quad (6.3)$$

Remark. Strictly speaking, the decomposition (6.3) is valid in the algebra A defined by the relation (5.7), but simultaneously the formula (6.3) shows that all the generators $\{h_\alpha, e_\alpha, T_\alpha^{\pm 1}, e_{\delta-\alpha}\}$ of the deformed Yangian $Y_{\eta,\xi}(sl_2)$ can be expressed in terms of the L-operator (5.3), what proves that the homomorphism constructed in Theorem 5.1 is an epimorphism.

The Gauss decomposition (6.3) provides the following choice of generators for $Y_{\eta,\xi}(sl_2)$:

$$\tilde{h}_\alpha^+(u) = T_\alpha^{-\frac{1}{2}} h_\alpha^+(u) T_\alpha^{-\frac{1}{2}} \quad , \quad \tilde{e}_\alpha^+(u) = e_\alpha^+(u) - \xi h_\alpha \quad , \quad \tilde{e}_{-\alpha}^+(u) = T_\alpha^{-\frac{1}{2}} e_{-\alpha}^+(u) T_\alpha^{-\frac{1}{2}} \quad . \quad (6.4)$$

Using relations (4.4)-(4.10) one can obtain the following relations between $\tilde{h}_\alpha(u)$, $\tilde{e}_{\pm\alpha}(u)$ and $T_\alpha^{\pm\frac{1}{2}}$:

$$T_\alpha^{\pm\frac{1}{2}} \tilde{h}_\alpha^+(u) T_\alpha^{\mp\frac{1}{2}} = \left(1 \pm \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{\mp 1} \tilde{h}_\alpha^+(u) \left(1 \pm \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1}, \quad (6.5)$$

$$T_\alpha^{\pm 1} \tilde{e}_\alpha^+(u) T_\alpha^{\mp 1} = \tilde{e}_\alpha^+(u) \mp 2\eta \xi \pm 2\eta \xi \left(1 \pm \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{\pm 1} \tilde{h}_\alpha^+(u) \left(1 \pm \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1}, \quad (6.6)$$

$$T_\alpha^{\pm\frac{1}{2}} \tilde{e}_{-\alpha}(u) T_\alpha^{\mp\frac{1}{2}} = \tilde{e}_{-\alpha}(u) \left(1 \pm \eta \xi \tilde{e}_{-\alpha}(u)\right)^{-1}. \quad (6.7)$$

With the help of the relations (4.4)-(4.10) and (6.5)-(6.7) we can prove the following theorem.

Theorem 6.1 *Defining relations for the "fields" $\tilde{h}_\alpha^+(u)$, $\tilde{e}_{\pm\alpha}^+(u)$ have the form:*

$$\mathcal{H}_1(u) \tilde{h}_\alpha^+(v) = \mathcal{H}_1(v) \tilde{h}_\alpha^+(u) \quad , \quad (6.8)$$

$$[\tilde{e}_\alpha^+(u), \tilde{e}_\alpha^+(v)] = \frac{\eta \left(\tilde{e}_\alpha^+(v) - \tilde{e}_\alpha^+(u) \right)^2}{v - u} + 2\eta \xi \left(\tilde{e}_\alpha^+(u) - \tilde{e}_\alpha^+(v) \right) \quad , \quad (6.9)$$

$$(u - v + \eta) \mathcal{G}_1(u) \tilde{e}_{-\alpha}^+(v) - (u - v - \eta) \mathcal{G}_1(v) \tilde{e}_{-\alpha}^+(u) = \eta \left(\mathcal{G}_1(u) \tilde{e}_{-\alpha}^+(u) + \mathcal{G}_1(v) \tilde{e}_{-\alpha}^+(v) \right) \quad , \quad (6.10)$$

$$(u - v - \eta) \mathcal{H}_1(u) \tilde{e}_{-\alpha}^+(v) - (u - v + \eta) \mathcal{G}_1(v) \tilde{h}_\alpha^+(u) = \eta \mathcal{H}_1(v) \tilde{e}_{-\alpha}^+(u) + \eta \mathcal{G}_1(u) \tilde{h}_\alpha(v) \quad , \quad (6.11)$$

$$\left(\tilde{e}_\alpha^+(u) - 2\eta \xi + 2\eta \xi \mathcal{H}_2(u) \right) \mathcal{G}_2(v) - \mathcal{G}_2(v) \tilde{e}_\alpha^+(u) = - \frac{\eta \left(\mathcal{H}_2(u) - \mathcal{H}_2(v) \right)}{u - v} + 2\eta \xi \mathcal{G}_2(v) \quad , \quad (6.12)$$

$$\begin{aligned} (u - v + \eta) \left(\tilde{e}_\alpha^+(v) - 2\eta \xi + 2\eta \xi \mathcal{H}_2(v) \right) \mathcal{H}_2(u) - (u - v - \eta) \mathcal{H}_2(u) = \\ = \eta \mathcal{H}_2(v) \tilde{e}_\alpha^+(u) + \eta \left(\tilde{e}_\alpha^+(u) - 2\eta \xi + 2\eta \xi \mathcal{H}_2(u) \right) \mathcal{H}_2(u) \quad . \end{aligned} \quad (6.13)$$

where

$$\mathcal{H}_1(u) = \left(1 - 2\eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \tilde{h}_\alpha^+(u) \left(1 - 2\eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \quad , \quad (6.14)$$

$$\mathcal{H}_2(u) = \left(1 - \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \tilde{h}_\alpha^+(u) \left(1 - \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \quad , \quad (6.15)$$

$$\mathcal{G}_1(u) = \left(1 - 2\eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \tilde{e}_{-\alpha}^+(u) \quad , \quad \mathcal{G}_2(u) = \left(1 - \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1} \tilde{e}_{-\alpha}^+(u) \quad . \quad (6.16)$$

Proof. Let us prove for instance the formula 6.10). From (4.9) we have

$$(u - v + \eta) e_{-\alpha}^+(u) e_{-\alpha}^+(v) - (u - v - \eta) e_{-\alpha}^+(v) e_{-\alpha}^+(u) = \eta (e_{-\alpha}^+(u))^2 + \eta (e_{-\alpha}^+(v))^2 \quad .$$

Substituting $e_{-\alpha}(u) = T_\alpha^{\frac{1}{2}} \tilde{e}_{-\alpha}^+(u) T_\alpha^{\frac{1}{2}}$ and using that $T_\alpha^{-\frac{1}{2}} \tilde{e}_{-\alpha}^+(u) T_\alpha^{\frac{1}{2}} = \tilde{e}_{-\alpha}^+(u) \left(1 - \eta \xi \tilde{e}_{-\alpha}^+(u)\right)^{-1}$ (see (6.7), we obtain the formula (6.10).

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